General solution of the diffusion equation with a nonlocal diffusive term and a linear force term

L. C. Malacarne,[1] R. S. Mendes,[1] E. K. Lenzi,[1] and M. K. Lenzi[2]
[1]*Universidade Estadual de Maringá, Departamento de Física, 87020-900 Maringá, Paraná, Brazil*
[2]*Departamento de Engenharia Química, Universidade Federal do Paraná, Setor de Tecnologia, Jardim das Américas, Caixa Postal 19011, 81531-990 Curitiba, Paraná, Brazil*
(Received 19 June 2006; published 17 October 2006)

We obtain a formal solution for a large class of diffusion equations with a spatial kernel dependence in the diffusive term. The presence of this kernel represents a nonlocal dependence of the diffusive process and, by a suitable choice, it has the spatial fractional diffusion equations as a particular case. We also consider the presence of a linear external force and source terms. In addition, we show that a rich class of anomalous diffusion, e.g., the Lévy superdiffusion, can be obtained by an appropriated choice of kernel.

DOI: 10.1103/PhysRevE.74.042101 PACS number(s): 05.40.Fb, 66.10.Cb, 05.20.−y, 05.60.−k

I. INTRODUCTION

The diffusive process is one of the most usual processes in nature and, since the Brown study and Einstein's first explanation [1], it has attracted attention in all science fields. In the last decades, diffusive processes that do not present the usual asymptotic time dependence on the second moment, i.e., $\langle x^2\rangle \sim t$, have also been related to a large class of physical systems. Illustrative examples are fluid transport in porous media [2], diffusion in plasmas [3], substance transported in a solvent from one vessel to another across a thin membrane [4], asymmetry of DNA translocation [5], relative diffusion in turbulent media [6], cetyltrimethylammonium bromide (CTAB) micelles dissolved in salted water [7], surface growth and transport of fluid in porous media [8], two dimensional rotating flow [9], subrecoil laser cooling [10], diffusion on fractals [11], anomalous diffusion at liquid surfaces [12], enhanced diffusion in active intracellular transport [13], particle diffusion in a quasi-two-dimensional bacterial bath [14], and spatiotemporal scaling of solar surface flows [15]. Thus the existence of the anomalous diffusion and its ubiquity has motivated the study of several approaches, in particular, the ones based on fractional diffusion equations [16–19] that have intensively been investigated. In fact, in Ref. [20] the fractional diffusion and wave equations are discussed, in [21] the boundary value problems for fractional diffusion equations are studied, in Ref. [22] a fractional Fokker-Planck equation is derived from a generalized master equation, in Ref. [23] the behavior of fractional diffusion at the origin is analyzed, in Ref. [24] a harmonic analysis of random fractional diffusion-wave equations is done, in Ref. [25] a fractional Kramers equation is introduced, and in Refs. [26–34] the solutions of the time-fractional diffusion equations are obtained.

In this Brief Report, we consider the formal solution of a large class of anomalous diffusion processes described by the equation

$$\frac{\partial}{\partial t}\hat{P}(x,t) = \frac{\mathcal{D}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathcal{K}(x-x')\frac{\partial^2}{\partial x'^2}\hat{P}(x',t)dx' - \frac{\partial}{\partial x}[F(x,t)\hat{P}(x,t)] + \alpha(t)\hat{P}(x,t), \qquad (1)$$

where $\mathcal{K}(x)$ is the kernel which contains a nonlocal dependence, $\mathcal{D}$ is the diffusion coefficient, $F(x,t)$ is the external force, and $\alpha(t)$ is a time-dependent source. Note that, due to the broadness of Eq. (1), it encompasses several scenarios of physical interest such as the distributed fractional diffusion equations [35], truncated Lévy flights [36], and advection-dispersion equations with a fractional Laplacian operator taking a general directional mixing measure into account [37]. In this direction, Eq. (1) may be used to investigate turbulence [38], anomalous diffusion in disordered media [39], and transport in the direction of flow in an aquifer with heavy tailed distribution [40].

In this paper, we work out Eq. (1) by taking a general kernel into account with $F(x,t)=-\mathcal{C}x$. In particular, we show how to obtain the generalized solution from the usual one for the case characterized by the absence of external force. We also discuss particular cases which emerge from choices $\mathcal{K}(x)\propto 1/|x|^{1+\mu}$ and $\mathcal{K}(x)\propto e^{-a|x|}$. These developments are presented in Sec. II and in Sec. III, we present our conclusions.

II. DIFFUSION EQUATION

Let us start our analysis by considering Eq. (1) without external force and subject to the initial condition $\hat{P}(x,0)=\sqrt{2\pi}\delta(x)$ and the boundary condition $\hat{P}(\pm\infty,t)=0$. In order to eliminate the source term of Eq. (1), we use the change $\hat{P}(x,t)=\exp[\int_0^t\alpha(t)dt]P(x,t)$, which leads us to the equation

$$\frac{\partial}{\partial t}P(x,t) = \frac{\mathcal{D}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathcal{K}(x-x')\frac{\partial^2}{\partial x'^2}P(x',t)dx'. \qquad (2)$$

Note that the above equation reduces to the usual diffusion equation for the kernel $\mathcal{K}(x)=\sqrt{2\pi}\delta(x)$ and other kernels imply a spatially nonlocal correlation. In addition, direct investigation shows that this diffusion equation can be correlated with a continuous time random walk for a Poissonian waiting time probability density function and a jump length probability density function λ given by $\tilde{\lambda}(k)=1-\tau\mathcal{D}k^2\tilde{\mathcal{K}}(k)$, where τ is the characteristic waiting time and $\tilde{\mathcal{K}}(k)$ is the Fourier transform ($\mathcal{F}\{\cdots\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dxe^{ikx}\cdots$ and $\mathcal{F}^{-1}\{\cdots\}=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dke^{-ikx}\cdots$) of $\mathcal{K}(x)$. Thus the presence of this kernel in Eq. (2) changes only the jump length probability density which; depending on the choice, may lead us to a distribution with long tail or compact behavior. In particular, for the first case, i.e., the long tail behavior, we may relate the distribution obtained with a Lévy distribution.

The solutions of Eq. (2) can lead us to cumbersome calculations depending on the choice of the kernel. However, it is possible to obtain them for a general kernel in terms of the solutions of the usual diffusion equation by using the Fourier transform. To show this feature, we consider the usual diffusion equation $\partial_t \mathcal{P}(x,t) = D\partial_x^2 \mathcal{P}(x,t)$, subject to the same boundary and initial conditions of Eq. (2), and the integral decomposition

$$P(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathcal{P}(x',t) W(x',x) dx'. \tag{3}$$

Now, in order to connect Eq. (3) with the solutions of Eq. (2), we choose $W(x',x)$ related to the kernel $\mathcal{K}(x)$ in the Fourier space as follows: $\widetilde{W}(x',k) = \exp[ix'k\sqrt{\widetilde{\mathcal{K}}(k)}]$, where $\widetilde{\mathcal{K}}(k)$ is the Fourier transform of $\mathcal{K}(x)$, as mentioned above. Thus we have the solution of Eq. (2) given in terms of the solution of the usual diffusion equation $\mathcal{P}(x,t)$.

To prove this statement, we can use the Fourier transform of $P(x,t)$, $\widetilde{P}(k,t)$, and employ the above $\widetilde{W}(x',k)$ leading us to the following result:

$$\begin{aligned}\widetilde{P}(k,t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathcal{P}(x',t) \widetilde{W}(x',k) dx' \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widetilde{\mathcal{P}}(x',t) \exp[ix'k\sqrt{\widetilde{\mathcal{K}}(k)}] dx' = \widetilde{\mathcal{P}}(k\sqrt{\widetilde{\mathcal{K}}(k)},t),\end{aligned} \tag{4}$$

where $\widetilde{\mathcal{P}}(k,t)$ is the spatial Fourier transform of $\mathcal{P}(x,t)$. On the other hand, from the convolution theorem, Eq. (2) becomes $\partial_t \widetilde{P}(k,t) = -\mathcal{D}k^2 \widetilde{\mathcal{K}}(k) \widetilde{P}(k,t)$. Now, considering the scaling of the distribution in the Fourier space, Eq. (4), and introducing the variable $w = k\sqrt{\widetilde{\mathcal{K}}(k)}$, the previous equation reduces to $\partial_t \widetilde{\mathcal{P}}(w,t) = -\mathcal{D}w^2 \widetilde{\mathcal{P}}(w,t)$, which is the usual diffusion equation in the Fourier space. This means that the solution of Eq. (2) in the Fourier space can be connected, as we discussed above, to the solution of the usual diffusion equation, $\widetilde{\mathcal{P}}(k,t)$, by the relation $\widetilde{P}(k,t) = \widetilde{\mathcal{P}}(k\sqrt{\widetilde{\mathcal{K}}(k)},t)$ and in the coordinate space by using Eq. (3). Similar analysis was performed in Ref. [41] for a time dependent kernel and in Ref. [42] for a fractional time diffusion equation by employing the Laplace transform.

An interesting result for the density distribution is obtained from the previous developments by considering the kernel $\widetilde{\mathcal{K}}(k) = -|k|^{\mu}$ with $-2 < \mu < 0$. In fact, this choice of kernel leads us to obtain as a solution of Eq. (2) the Lévy distributions, i.e.,

$$P(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-|k|^{2+\mu}\mathcal{D}t} e^{-ikx} dk, \tag{5}$$

which enables us to relate it to a continuous random walk with long jumps described by fractional diffusion equations which employ spatial derivatives. By performing the above integral, the solution $P(x,t)$ is given in terms of the Fox H function [43] as follows:

$$P(x,t) = \frac{1}{(2+\mu)|x|} \mathrm{H}_{2,2}^{1,1}\left[\frac{|x|}{(\mathcal{D}t)^{1/(2+\mu)}} \middle|\begin{matrix} (1,1/(2+\mu)),(1,1/2) \\ (1,1),(1,1/2) \end{matrix} \right]. \tag{6}$$

In the presence of an external linear force term, $F(x,t) = -\mathcal{C}x$, we cannot connect the solution of Eq. (1) to the solution of the usual diffusion equation. But in this case, for the initial condition, $P(x,0) = \sqrt{2\pi}\delta(x)$, we get the formal solution in the Fourier space, $\widetilde{P}(k,t)$,

$$\widetilde{P}(k,t) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{\mathcal{D}}{C} \int_{ke^{-Ct}}^{k} u\widetilde{\mathcal{K}}(u) du \right]. \tag{7}$$

To show this result, note that in the Fourier space Eq. (2) with linear force becomes

$$\frac{\partial}{\partial t} \widetilde{P}(k,t) = -\mathcal{D}k^2 \widetilde{\mathcal{K}}(k) \widetilde{P}(k,t) - Ck \frac{\partial}{\partial k} \widetilde{P}(k,t). \tag{8}$$

Now writing the solution as

$$\widetilde{P}(k,t) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{\mathcal{D}}{C} \int^{k} u\widetilde{\mathcal{K}}(u) du \right] \widetilde{\rho}(k,t), \tag{9}$$

Eq. (8) reduces to

$$\frac{\partial}{\partial t} \widetilde{\rho}(k,t) = -Ck \frac{\partial}{\partial k} \widetilde{\rho}(k,t), \tag{10}$$

with the initial condition $\widetilde{\rho}(k,0) = \exp\left[\frac{\mathcal{D}}{C}\int^k u\widetilde{\mathcal{K}}(u)du\right]$. To satisfy this initial condition, we can take the solution of $\widetilde{\rho}(k,t)$ as

$$\widetilde{\rho}(k,t) = \exp\left[\frac{\mathcal{D}}{C} \int^{\widetilde{f}(k,t)} u\widetilde{\mathcal{K}}(u) du \right], \tag{11}$$

with $\widetilde{f}(k,0) = k$. Substituting Eq. (11) in Eq. (10), we get that $\widetilde{f}(k,t)$ obeys the equation $\partial_t \widetilde{f}(k,t) = -Ck\partial_k \widetilde{f}(k,t)$, which presents the simple solution $\widetilde{f}(k,t) = ke^{-Ct}$. Using this solution in Eq. (11) together with Eq. (9), we get the solution Eq. (7).

Now, let us discuss four special cases of Eq. (7) with $\mathcal{C} > 0$. The first one is the stationary case which can be obtained from Eq. (7) by taking the asymptotic limit $t \to \infty$ into account. In particular, for this case, the stationary solution $\widetilde{P}_{st}(k)$ is

$$\widetilde{P}_{st}(k) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{\mathcal{D}}{C} \int_{0}^{k} u\widetilde{\mathcal{K}}(u) du \right]. \tag{12}$$

Notice that, depending on the choice of the kernel $\widetilde{\mathcal{K}}(k)$ in Eq. (12), we may have, for example, a long tail behavior characterized by a power-law behavior in the asymptotic limit or a short tail behavior given in terms of exponentials. In particular, the solutions which present an asymptotic behavior like a power law may be related to the Lévy distributions. This fact suggests a thermostatics context different from the usual one, in contrast to the analysis performed in Ref. [48] for the time fractional diffusion equations. The second case is $\tilde{\mathcal{K}}(k)=1$ which recovers the usual Ornstein-Uhlenbeck process [44],

$$\tilde{P}(k,t) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{\mathcal{D}}{2C}(1-e^{-2Ct})k^2 \right], \qquad (13)$$

where its second derivative in k leads to the second moment $\langle x^2(t)\rangle = \mathcal{D}(1-e^{-2Ct})/\mathcal{C}$. The third case is given by the kernel $\tilde{\mathcal{K}}(k) = -|k|^{\mu}$ that leads us to obtain

$$\tilde{P}(k,t) = \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{\mathcal{D}|k|^{\mu+2}}{(\mu+2)C}(1-e^{-(\mu+2)Ct}) \right]. \qquad (14)$$

This equation is a generalization of the Ornstein-Uhlenbeck process usually employed to investigate systems which present anomalous diffusion [45]. Note that Eq. (5), which corresponds to the free-force fractional diffusion equation, is obtained from the above equation in the limit $C \to 0$. Another feature concerning Eq. (14) is that the second moment is not defined in the range of the parameter μ where the distribution is stable. For the fourth case, we analyze a kernel which presents an exponential decay, i.e., $\mathcal{K}(x) = \sqrt{\pi/2}ae^{-a|x|}$ with $a>0$. This kernel represents a short range spatial interaction characterized by the fast decay of the exponential function. By substituting the Fourier transform of $\mathcal{K}(x)$ $[\tilde{\mathcal{K}}(k) = a^2/(a^2+k^2)]$ in Eq. (7) and performing an integration, we obtain

$$\tilde{P}(k,t) = \frac{1}{\sqrt{2\pi}} \left[1 - \frac{k^2(1-e^{-2Ct})}{a^2+k^2} \right]^{\mathcal{D}a^2/2C}. \qquad (15)$$

From this equation, we can recover the Ornstein-Uhlenbeck process, Eq. (13), by taking the limit $a\to\infty$ and obtain the free-force case for this kernel in the limit $C\to 0$. In addition, due to the short range behavior of the kernel, we obtain an interesting result to the second moment of the non-Gaussian distribution given by Eq. (15). It is exactly the same as the one previously obtained for the Ornstein-Uhlenbeck process. This fact illustrates that the normal diffusion may not be associated with the Gaussian shape of the distribution. The same possibility was explored in Refs. [46,47] when dealing with nonlinear diffusion equations with spatial and temporal dependence on the coefficients. The stationary solution associated to Eq. (15) in the real space is

$$P(x) = \{a/[\sqrt{\pi}\Gamma(a^2\mathcal{D}/(2\mathcal{C}))]\}2/(a|x|)^{\nu}K_{\nu}(a|x|),$$

with $\nu = 1/2 - a^2\mathcal{D}/(2\mathcal{C})$ where $K_\nu(x)$ is a modified Bessel function.

In Fig. 1, we show the stationary distribution for particles in the harmonic potential related to the three kernels used: (i) the delta kernel, corresponding to the usual Gaussian diffusion; (ii) the power law kernel, corresponding to Lévy like diffusion; and (iii) the exponential kernel, corresponding to the inverse Fourier transform of Eq. (15). We have chosen typical values of parameters $\mathcal{C}$, a, and μ which lead us to different forms for the distributions.

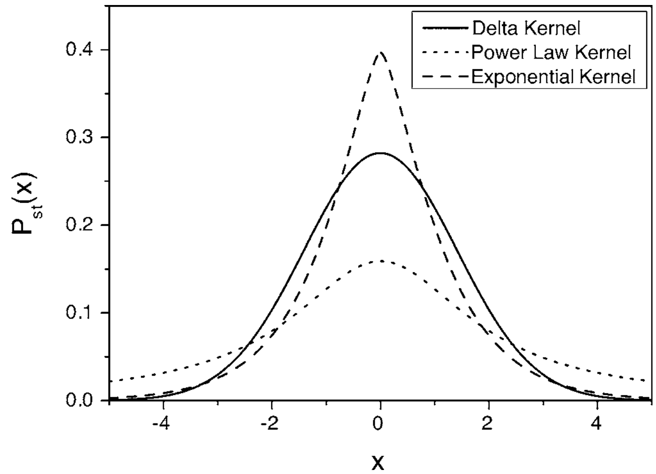

FIG. 1. Stationary distribution for the diffusion in the harmonic potential corresponding to the usual Gaussian diffusion (solid line), the distribution with a power law kernel (dotted line), and the distribution with an exponential kernel (dashed line). The parameters used were $\mathcal{C}=0.5$, $\mu=-0.9$, and $a=1$.

III. SUMMARY AND CONCLUSION

In summary, we have investigated a general nonlocal diffusion equation by considering the presence of a linear external force and a time dependent source term. We show that it is possible to connect the solution of this nonlocal equation to the solution of the usual diffusion equation in the absence of a drift term. In this manner, the solution of Eq. (1) for this case may be obtained by applying the inverse Fourier transform on the usual solution after performing suitable changes. In the presence of a drift term, we obtained a formal solution in the Fourier space for an arbitrary kernel. In particular, for this case, in the absence of a source term, the solution for long times presents a stationary distribution which depending on the choice of the kernel may have a short or a long tail behavior. In this context, we have also analyzed simple cases such as $\tilde{\mathcal{K}}(k) = -|k|^{\mu}$ and $\tilde{\mathcal{K}}(k) = a^2/(a^2+k^2)$. The first choice to the kernel leads us to the Lévy distributions which have the asymptotic behavior characterized by a power law behavior, i.e., $P \sim 1/|x|^{1+\bar{\mu}}$ and the second moment is not finite. For the second one, we got a usual time dependence for the second moment with a non-Gaussian distribution and the asymptotic behavior is given by $P \sim e^{-a|x|}/|x|^{1-\mathcal{D}a^2/2\mathcal{C}}$. Furthermore, the advantage of the formal solution is that, even if the analytical solution in the coordinates space is not known, a numerical treatment of the inverse Fourier transform can be easier than a direct solution of the differential equation.

Finally, we hope that the results obtained here may be useful to analyze a large class of different anomalous diffusive processes in specific theoretical and experimental contexts by the appropriate choice of the kernel.

ACKNOWLEDGMENT

We thank CNPq (Brazilian Agency) for partial financial support.

[1] A. Einstein, Ann. Phys. (Leipzig) **17**, 549 (1905).
[2] J. D. Seymour, J. P. Gage, S. L. Codd, and R. Gerlach, Phys. Rev. Lett. **93**, 198103 (2004).
[3] J. G. Berryman, J. Math. Phys. **18**, 2108 (1977).
[4] T. Kosztolowicz, K. Dworecki, and S. Mrowczynski, Phys. Rev. Lett. **94**, 170602 (2005).
[5] R. C. Lua and A. Y. Grosberg, Phys. Rev. E **72**, 061918 (2005).
[6] H. G. E. Hentschel and I. Procaccia, Phys. Rev. A **29**, 1461 (1984).
[7] A. Ott, J. P. Bouchaud, D. Langevin, and W. Urbach, Phys. Rev. Lett. **65**, 2201 (1990).
[8] H. Spohn, J. Phys. I **3**, 69 (1993).
[9] T. H. Solomon, E. R. Weeks, and H. L. Swinney, Phys. Rev. Lett. **71**, 3975 (1993).
[10] F. Bardou, J. P. Bouchaud, O. Emile, A. Aspect, and C. Cohen-Tannoudji, Phys. Rev. Lett. **72**, 203 (1994).
[11] H. E. Roman, Fractals **12**, 149 (2004); J. Stephenson, Physica A **222**, 234 (1995); R. Metzler, W. G. Glöckle, and T. F. Nonnenmacher, *ibid.* **211**, 13 (1994).
[12] O. V. Bychuk and B. O'Shaughnessy, Phys. Rev. Lett. **74**, 1795 (1995); S. C. Venkataramani, T. M. Antonsen, and E. Ott, *ibid.* **78**, 3864 (1997).
[13] A. Caspi, R. Granek, and M. Elbaum, Phys. Rev. Lett. **85**, 5655 (2000).
[14] Xiao-Lun Wu and A. Libchaber, Phys. Rev. Lett. **84**, 3017 (2000).
[15] J. K. Lawrence, A. C. Cadavid, A. Ruzmaikin, and T. E. Berger, Phys. Rev. Lett. **86**, 5894 (2001).
[16] R. Hilfer, *Applications of Fractional Calculus in Physics* (World Scientific, Singapore, 2000).
[17] R. Metzler and J. Klafter, Phys. Rep. **339**, 1 (2000).
[18] B. J. West, M. Bologna, and P. Grigolini, *Physics of Fractal Operators* (Springer, New York, 2002).
[19] G. M. Zaslavsky, Phys. Rep. **371**, 461 (2002).
[20] W. R. Schneider and W. Wyss, J. Math. Phys. **30**, 134 (1989).
[21] R. Metzler and J. Klafter, Physica A **278**, 107 (2000).
[22] R. Metzler, E. Barkai, and J. Klafter, Europhys. Lett. **46**, 431 (1999).
[23] Y. E. Ryabov, Phys. Rev. E **68**, 030102(R) (2003).
[24] V. V. Anh and N. N. Leonenko, Appl. Math. Comput. **141**, 77 (2003).
[25] E. Barkai and R. J. Silbey, J. Phys. Chem. B **104**, 3866 (2000).
[26] F. Mainardi and G. Pagnini, Appl. Math. Comput. **141**, 51 (2003).
[27] B. N. N. Achar and J. W. Hanneken, J. Mol. Liq. **114**, 147 (2004).
[28] Fu-Yao Ren, Jin-Rong Liang, Wei-Yuan Qiu, Xiao-Tian Wang, Y. Xu, and R. R. Nigmatullin, Phys. Lett. A **312**, 187 (2003).
[29] P. Agrawal, Nonlinear Dyn. **29**, 145 (2002).
[30] S. A. El-Wakil and M. A. Zahran, Chaos, Solitons Fractals **12**, 1929 (2001).
[31] S. A. El-Wakil, A. Elhanbaly, and M. A. Zahran, Chaos, Solitons Fractals **12**, 1035 (2001).
[32] E. K. Lenzi, R. S. Mendes, Kwok Sau Fa, L. C. Malacarne, and L. R. da Silva, J. Math. Phys. **44**, 2179 (2003).
[33] E. K. Lenzi, R. S. Mendes, J. S. Andrade, L. R. da Silva, and L. S. Lucena, Phys. Rev. E **71**, 052101 (2005).
[34] P. C. Assis, R. P. de Souza, P. C. da Silva, L. R. da Silva, L. S. Lucena, and E. K. Lenzi, Phys. Rev. E **73**, 032101 (2006).
[35] A. V. Chechkin, R. Gorenflo, and I. M. Sokolov, Phys. Rev. E **66**, 046129 (2002).
[36] I. M. Sokolov, A. V. Chechkin, and J. Klafter, Physica A **336**, 245 (2004).
[37] M. M. Meerschaert, D. A. Benson, and B. Baeumer, Phys. Rev. E **63**, 021112 (2001).
[38] M. F. Shlesinger, B. J. West, and J. Klafter, Phys. Rev. Lett. **58**, 1100 (1987); J. Klafter, A. Blumen, and M. F. Shlesinger, Phys. Rev. A **35**, 3081 (1987).
[39] J.-P. Bouchaud and A. Georges, Phys. Rep. **195**, 127 (1990).
[40] D. A. Benson, R. Schumer, S. W. Wheatcraft, and M. M. Meerschaert, Transp. Porous Media **42**, 211 (2001).
[41] I. M. Sokolov, Phys. Rev. E **66**, 041101 (2002).
[42] T. Kosztolowicz, J. Phys. A **37**, 10779 (2004).
[43] A. M. Mathai and R. K. Saxena, *The H-function with Application in Statistics and Other Disciplines* (Wiley Eastern, New Delhi, 1978).
[44] H. Risken, *The Fokker-Planck Equation, Methods of Solution and Applications* (Springer-Verlag, New York, 1984).
[45] S. Jespersen, R. Metzler, and H. C. Fogedby, Phys. Rev. E **59**, 2736 (1999).
[46] L. C. Malacarne, R. S. Mendes, I. T. Pedron, and E. K. Lenzi, Phys. Rev. E **65**, 052101 (2002).
[47] L. C. Malacarne, R. S. Mendes, I. T. Pedron, and E. K. Lenzi, Phys. Rev. E **63**, 030101(R) (2001).
[48] I. M. Sokolov, Phys. Rev. E **63**, 056111 (2001); I. M. Sokolov, J. Klafter, and A. Blumen, *ibid.* **64**, 021107 (2001).